

Thom polynomials for maps of curves with isolated singularities

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February 1, 2008

*To Vladimir Igorevich Arnold
on the occasion of his 70 birthday
with gratitude and admiration*

Abstract

Thom (residual) polynomials in characteristic classes are used in the analysis of geometry of functional spaces. They serve as a tool in description of classes Poincaré dual to subvarieties of functions of prescribed types. We give explicit universal expressions for residual polynomials in spaces of functions on complex curves having isolated singularities and multisingularities, in terms of few characteristic classes. These expressions lead to a partial explicit description of a stratification of Hurwitz spaces.

1 Introduction

In [1] V. I. Arnold investigated spaces of Laurent (or trigonometric) polynomials in one variable. Each such space is determined by a pair of positive integers, the orders of the poles. Extending earlier results by Looijenga [10] and Lyashko (see [2]) valid for polynomials he constructed a compactification of the space of Laurent polynomials and proved the $K(\pi, 1)$ -property for the subspace of nondegenerate Laurent polynomials (those whose all critical values are pairwise distinct). The Lyashko–Looijenga map taking each function to the set of its critical values extends to a polynomial finite map on Arnold’s compactification, and in [1] the degree of this map was computed.

Degenerate Laurent polynomials form the discriminant, which is stratified according to the degeneration types. In the present paper, we deduce formulas for the

*Steklov Mathematical Institute RAS and the Poncelet Laboratory, Independent University of Moscow, partly supported by the RFBR Grant 04-01-00762

[†]Institute for System Research RAS and the Poncelet Laboratory, Independent University of Moscow, partly supported by the grant ACI-NIM-2004-243 (Noëds et tresses)

[‡]Both authors are partly supported by the RFBR Grant 05-01-01012-a, NWO-RFBR 047.011.2004.026 (RFBR 05-02-89000-NWOa), GIMP ANR-05-BLAN-0029-01

degree of the Lyashko–Looijenga map restricted to so-called primitive strata, that is, strata of Laurent polynomials with a single finite degenerate value. For the case of polynomials the degrees of the restriction of the Lyashko–Looijenga map to all strata were computed in [8, 9, 14, 15]. In fact, our results are of much more general nature. They are based on the study of spaces of meromorphic functions on complex curves (Hurwitz spaces) initiated in [6]. The main tool of the study is the theory of universal residual polynomials in characteristic classes developed mainly by R. Thom [13] for singularities and Kazaryan [4] for multisingularities. In principle, this theory allows one to describe (the cohomology classes Poincaré dual to) strata in functional spaces formed by functions with singularities of prescribed types. However, explicit calculations of the residual polynomials often prove to be cumbersome.

In [6], our calculations were mainly based on the indefinite coefficients method efficiently applied to computing universal polynomials by Rimanyi [12]. Numerous calculations of this kind we made led to a number of conjectures concerning the explicit form of these polynomials. A part of them, concerning the strata of multisingularities in unfoldings of isolated singularities, we prove here.

A holomorphic function on a smooth complex curve can have isolated singularities only of type A_n , that is, those having the form $z \mapsto z^{n+1}$ in appropriate coordinates. In holomorphic families whose generic element is smooth, curves with double points arise in an unavoidable way. A typical example is the family of hyperbolas $xy = \varepsilon$ on the complex plane, which degenerates into the pair of coordinate axes at $\varepsilon = 0$, each being a branch of the curve. A function having an isolated singularity at the double point belongs to the type $I_{k,l}$ if its restriction to one branch at this point is of type A_{k-1} and to the other branch is of type A_{l-1} . The types A_n and $I_{k,l}$ are the only possible types of isolated singularities of functions on curves, and our goal will be the analysis of their universal unfoldings.

The universal polynomials for singularities are usually expressed in terms of the Chern classes of (the tangent bundles over) the manifolds under study. In particular, the number of basic classes grows as the complexity, whence the codimension, of the singularity grows. In the case of families of functions on curves, however, one can manage with finitely many basic classes, whatever is the codimension. In particular, if the functions in the family acquire only isolated singularities, then four basic classes are sufficient [6]. Specializing the universal polynomials in the four basic classes to spaces of versal unfoldings of isolated singularities one can obtain explicit formulas for double Hurwitz numbers.

2 Enumeration of Laurent polynomials with prescribed multisingularities

2.1 Stratification of standard versal unfoldings

Consider the following two families of rational functions:

$$x \mapsto x^{n+1} + a_2 x^{n-2} + \cdots + a_n x + a_{n+1} \tag{1}$$

and

$$xy = \varepsilon, \quad (x, y) \mapsto x^k + y^\ell + a_1 x^{k-1} + \cdots + a_{k-1} x + c + b_{\ell-1} y + \cdots + b_1 y^{\ell-1}. \quad (2)$$

The domain of a function in the first family is the complex line with coordinate x and the coefficients $(a_2, \dots, a_{n+1}) \in \mathbb{C}^n$ form the parameter space of the family. This family of polynomials is the standard versal deformation of the singularity A_n .

The second family is parameterized by the points $(\varepsilon, a_1, \dots, a_{k-1}, c, b_1, \dots, b_{\ell-1}) \in \mathbb{C}^{k+\ell}$. The domain of a function here is the curve $\{xy = \varepsilon\} \subset \mathbb{C}^2$. This curve is smooth for $\varepsilon \neq 0$ and has a double point for $\varepsilon = 0$. This family is the space of Laurent polynomials of bidegree (k, ℓ) , or the standard versal deformation of the singularity $I_{k,\ell}$.

The zeroes of the differential of a function are called its critical points. For generic parameter values the functions in both families have only simple (Morse) critical points with distinct finite critical values whose number is n and $k + \ell$, respectively. But for some parameter values the function has more complicated singularities. We say that a function f acquires a local singularity of type A_m at some point of the source if the function can be represented in the form $z \mapsto z^{m+1}$ for an appropriate choice of the local coordinates. A function acquires a multisingularity of type A_{m_1, \dots, m_r} at a point of the target if the preimage of this point contains pairwise distinct points where the function has local singularities of types A_{m_1}, \dots, A_{m_r} , respectively. The number of simple critical values that collapse at a point with multisingularity A_{m_1, \dots, m_r} is equal to $|m| = m_1 + \cdots + m_r$. Functions with multisingularity A_{m_1, \dots, m_r} form the stratum σ_{m_1, \dots, m_r} in the space of functions. The codimension of this stratum is $|m| - 1$.

For both families, we consider the following problem: *find the number of generic functions in the family that have critical value 0 with the multisingularity type A_{μ_1, \dots, μ_r} , with prescribed simple critical values.*

These numbers are special cases of what is called *double Hurwitz numbers*, and a variety of formulas for these numbers is known [11, 3]. However, our answer is represented in a different form and the equivalence of our formulas to the known ones is by no means evident.

Our approach is close to that of Arnold in [1]. To state it, let μ denote the number of parameters in the family. For the deformation of the singularity A_n it equals n , and for the singularity $I_{k,\ell}$ it is $k + \ell$. It is equal also to the number of critical points of a generic function in the family. The *Lyashko-Loojenga map* $\Lambda : \mathbb{C}^\mu \rightarrow \mathbb{C}^\mu$ associates to a parameter value the unordered tuple of critical values of the function. This definition is applied if the function has only simple critical points. However, Λ extends to a holomorphic (and even polynomial) proper quasihomogeneous map to the whole space \mathbb{C}^μ (see [1, 10]; explicit coordinate expressions for the Lyashko-Loojenga map can be found in [7], Chapter 5). What we are interested in is actually the degree of the restriction of Λ to a particular multisingularity stratum. The degree of Λ on the whole parameter space can easily be computed as the ratio of the products of coordinate weights in the target and the source, which yields $(n+1)^{n-1}/(n+1)!$ for the case of polynomials and $k^k \ell^\ell / (k! \ell!)$ for Laurent polynomials.

In the case of polynomials each multisingularity stratum admits an explicitly given

nonsingular normalization. This allows one to apply the same quasihomogeneity argument in order to compute the corresponding degree.

Theorem 2.1 ([8]) *In the case of polynomials, the degree of the restriction of the Lyashko–Looijenga map to the multisingularity stratum, which is the closure of functions having multisingularity of type m_1, \dots, m_r , is*

$$(n+1)^{n-1-|m|} \frac{(n-|m|)!}{|\text{Aut}(m_1, \dots, m_r)|(n-r-|m|)!},$$

where $|\text{Aut}(m_1, \dots, m_r)|(n-r-|m|)!$ denotes the order of the automorphism group of the tuple m_1, \dots, m_r , that is, the product of the factorials of numbers of coinciding elements in this tuple.

Below, we compute the A -contribution to universal formulas using the tools exploited to obtain this result.

In the case of Laurent polynomials the strata have no good parametrization. To overcome this difficulty, instead of the direct geometric study of the stratum we relate the degree of the restriction of Λ to that stratum to the degree of Λ on the whole space. The most convenient language to establish a relationship between these degrees is that of *equivariant cohomology*.

The multiplicative group \mathbb{C}^* of nonzero complex numbers acts on the spaces of standard versal unfoldings of the singularities A_n and $I_{k,\ell}$ by multiplication by an appropriate power of the complex number. For an element $\lambda \in \mathbb{C}^*$, this action is

$$\lambda : (x, a_2, \dots, a_{m+1}) \mapsto (\lambda x, \lambda^2 a_2, \dots, \lambda^{n+1} a_{n+1}) \quad (3)$$

for the unfolding of the singularity A_n and

$$\begin{aligned} & \lambda : (x, y, \varepsilon, a_1, \dots, a_{k-1}, c, b_1, \dots, b_{\ell-1}) \\ & \mapsto (\lambda^\ell x, \lambda^k y, \lambda^{k+\ell} \varepsilon, \lambda^\ell a_1, \dots, \lambda^{(k-1)\ell} a_{k-1}, \lambda^{k\ell} c, \lambda^k b_1, \dots, \lambda^{(\ell-1)k} b_{\ell-1}) \end{aligned} \quad (4)$$

for the unfolding of the singularity $I_{k,\ell}$. The restriction of this action to the coordinate space a_i (respectively, ε, a_i, c, b_j) defines an action of \mathbb{C}^* on the parameter space \mathbb{C}^μ . Denote by $H_{\mathbb{C}^*}^*(\mathbb{C}^\mu)$ the \mathbb{C}^* -equivariant cohomology of the parameter space \mathbb{C}^μ with respect to this action. Since \mathbb{C}^μ is contractible, we have $H_{\mathbb{C}^*}^*(\mathbb{C}^\mu) \simeq H_{\mathbb{C}^*}^*(\text{pt}) \simeq \mathbb{Q}[\tau]$, where $\tau = c_1(\mathcal{O}(1)) \in H_{\mathbb{C}^*}^2(\mathbb{C}^\mu)$ is the standard characteristic class of the \mathbb{C}^* -action under consideration (here and below we work with cohomology with rational coefficients). We also make use of the similar cohomology rings of the space of unfolding and its image (both isomorphic to $\mathbb{C}^{\mu+1}$). Each of these rings is a ring of polynomials in one variable $\tau = c_1(\mathcal{O}(1))$, and in order to distinguish between elements of these rings we will sometimes make use of notation τ_X, τ_Y and τ_B for the generators in the equivariant cohomology rings of the source space (the space of the unfolding), the target space and the base (the space of the deformation), respectively.

Consider the stratum of a multisingularity $\sigma = \sigma_{m_1, \dots, m_r} \subset \mathbb{C}^\mu$. The group \mathbb{C}^* preserves this stratum, and its projectivization $P\sigma = (\sigma - \{0\})/\mathbb{C}^*$ is a compact

algebraic variety endowed with a natural fundamental homology class. The number

$$\deg \sigma = \int_{P\sigma} \tau^{\dim P\sigma}$$

is called the *degree* of the stratum. Moreover, the equivariant cohomology class Poincaré dual to this subvariety $\sigma \subset \mathbb{C}^\mu$ is well defined. By definition, this class is proportional to $\tau^{|m|-1}$, where $|m| - 1 = \text{codim } \sigma = \sum m_i - 1$. The proportionality coefficient can be easily expressed in terms of the degrees of the stratum and the ambient space:

$$[\sigma] = \frac{\deg \sigma}{\deg \mathbb{C}^\mu} \tau^{|m|-1}.$$

The degree of σ and of its image, as well as the degree of the mapping Λ and its restriction $\Lambda|_\sigma$ to σ , are related by the following natural equations that reduce computations of the degree of the map to that of the degrees of geometric objects:

Proposition 2.2 *The degree of the restriction of Λ to a stratum σ , the degree $\deg \sigma$ of this stratum and the cohomology class $[\sigma]$ dual to $[\sigma]$ are subject to the equations*

$$\deg \Lambda|_\sigma = \frac{\deg \sigma}{\deg \Lambda(\sigma)} = \frac{[\sigma]}{[\Lambda(\sigma)]} \frac{\deg \mathbb{C}^\mu}{\deg \Lambda(\mathbb{C}^\mu)} = \frac{[\sigma]}{[\Lambda(\sigma)]} \deg \Lambda.$$

In particular, the generic stratum, which coincides with the entire deformation space \mathbb{C}^μ , in the case A_n has the degree $1/(n+1)!$, while the degree of its image under Λ is $1/((n+1)^n n!)$, and we obtain $(n+1)^{n-1}$ for the degree of Λ on this stratum. For the deformation of the singularity $I_{k,\ell}$, the degree of the generic stratum is inverse to the product of the weights of all variables, i.e., it is $1/((k+\ell)k!\ell!k^{\ell-1}\ell^{k-1})$, while the degree of the image is $1/((k+\ell)!(k\ell)^{k+\ell})$, which yields

$$\deg \Lambda = (k+\ell-1)! \frac{k^{k+1}\ell^{\ell+1}}{k!\ell!}$$

for the degree of Λ .

2.2 The degrees of primitive strata in the standard unfolding of $I_{k,\ell}$

Theorem 3.5 in Sec. 3 of the present paper gives universal formulas for the cohomology classes of primitive strata in the versal unfolding of the singularity $I_{k,\ell}$. Here we show how these formulas can be used for computing the degrees of primitive strata for given orders k and ℓ of the poles. We restrict ourselves with the codimension 1 strata, the *caustic*, which is formed by functions with multiple critical points, and the *Maxwell stratum*, which consists of functions having coinciding critical values at distinct critical points. For strata of bigger codimension, the computations are similar.

The caustic $\sigma_2 \subset B = \mathbb{C}^\mu$ is nothing but the image of the projection of the stratum $A_2(\mathbb{C}^{\mu+1}) \subset X = \mathbb{C}^{\mu+1}$ to the base. Theorem 3.5 yields for the I -contribution to the dual class the universal expression

$$A_2(X) = \psi^2 - \nu_1 \nu_2.$$

In order to evaluate this expression in the unfolding of the singularity $I_{k,\ell}$, one must make the substitution $\psi = k\ell\tau_X$, $\nu_1 = k\tau_X$, $\nu_2 = \ell\tau_X$, whence

$$A_2(\mathbb{C}^{\mu+1}) = ((k\ell)^2 - k\ell)\tau_X^2.$$

Note that for $k = \ell = 1$, this expression is 0, in accordance with the fact that the caustic in the versal deformation of $I_{1,1}$ is empty.

The map p is the projection to the parameter space. The action of the Gysin homomorphism p_* on the generator τ_X^d of the $2d$ -cohomology has the form $p_*\tau_X^d = \frac{k+\ell}{k\ell}\tau_B^{d-1}$, and we obtain the following expression for the caustic in the space $B = \mathbb{C}^\mu$:

$$\sigma_2 = (k + \ell)(k\ell - 1)\tau_B,$$

whence the degree of the caustic is the degree of the generic stratum times $(k + \ell)(k\ell - 1)$, i.e.,

$$\deg \sigma_2 = \frac{k\ell - 1}{k!\ell!k^{\ell-1}\ell^{k-1}}.$$

Dividing this by the degree $1/((k + \ell - 2)!(k\ell)^{k+\ell-1})$ of the image, we obtain as the result the degree of the restriction of Λ to the caustic:

$$\deg \Lambda|_{\sigma_2} = (k + \ell - 2)! \frac{k^k \ell^\ell}{k!\ell!} (k\ell - 1).$$

For the Maxwell stratum, which is a subvariety in the image of the unfolding, the universal formula for the I -contribution looks like

$$A_{1,1}(Y) = \frac{\psi(\nu_1 + \nu_2)}{2\nu_1^2\nu_2^2}(\psi^3(\nu_1 + \nu_2) - 4\psi^2\nu_1\nu_2 + 2\nu_1^2\nu_2^2).$$

Substituting $\psi = k\ell\tau_Y$, $\nu_1 = k\tau_Y$, $\nu_2 = \ell\tau_Y$, we obtain the expression for this class in the unfolding of the deformation of $I_{k,\ell}$,

$$A_{1,1}(Y) = \frac{1}{2}k\ell(k + \ell)(k\ell(k + \ell) - 4k\ell + 2)\tau_Y^2.$$

This expression vanishes both for $k = \ell = 1$ and for $k = 2, \ell = 1$ or $k = 1, \ell = 2$: the Maxwell stratum is empty in the deformations of the singularities $I_{1,1}$ and $I_{1,2}$.

Now, the application of the Gysin homomorphism q_* yields

$$\sigma_{1,1} = q_*(A_{1,1}(Y)) = \frac{1}{2}(k + \ell)(k\ell(k + \ell) - 4k\ell + 2)\tau_B,$$

which gives the following expressions for the degree of the stratum $\sigma_{1,1}$ and the restriction of Λ to this stratum:

$$\begin{aligned} \deg \sigma_{1,1} &= \frac{k\ell(k + \ell) - 4k\ell + 2}{k!\ell!k^{\ell-1}\ell^{k-1}}, \\ \deg \Lambda|_{\sigma_{1,1}} &= (k + \ell - 2)! \frac{k^k \ell^\ell}{k!\ell!} \frac{1}{2} (k\ell(k + \ell) - 4k\ell + 2). \end{aligned}$$

Remark 2.3 Note that, similarly to the case of the Maxwell stratum, one could seek for the expression of the caustic starting with the stratum $A_2(Y)$ in the target space, the result, of course, would be the same.

In the case of a general stratum σ_{m_1, \dots, m_r} , the degree of the restriction of Λ to this stratum has the form

$$\deg \Lambda|_{\sigma_{m_1, \dots, m_r}} = (k + \ell - |m|)! \frac{k^{k+2-|m|} \ell^{\ell+2-|m|}}{k! \ell!} P_{m_1, \dots, m_r},$$

where P_{m_1, \dots, m_r} is a symmetric polynomial in k and ℓ . For strata of small codimension, this polynomial is given by the table below.

(m_1, \dots, m_r)	P_{m_1, \dots, m_r}
(2)	$k\ell - 1$
(1, 1)	$\frac{1}{2}(k\ell(k + \ell) - 4k\ell + 2)$
(3)	$(k\ell)^2 - 5k\ell + 2(k + \ell)$
(2, 1)	$(k\ell - 3)(k\ell(k + \ell) - 6k\ell + 2(k + \ell))$
(1, 1, 1)	$\frac{1}{6}((k\ell)^2((k + \ell)^2 - 12(k + \ell) + 40) + k\ell(6(k + \ell) - 80) + 24(k + \ell))$

Note that in order to compute a double Hurwitz number, one has to divide the degree of Λ restricted to the corresponding stratum by $k\ell |\text{Aut}(k, \ell)|$. Here the product of k and ℓ corresponds to the fact that each equivalence class of a Laurent polynomial has exactly $k\ell$ representatives in the standard versal deformation of $I_{k, \ell}$ (see [1]). The order of the automorphism group is either 1 or 2 depending on whether the orders k and ℓ of the poles coincide or not; if they coincide, then the transposition of the poles defines an automorphism of the unfolding.

3 Universal formulas

3.1 Relative characteristic classes and universal polynomials

The subject of our study are commutative triangles of spaces and maps of the form

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ & \searrow \tilde{p} & \swarrow \tilde{q} \\ & B & \end{array}$$

Here \tilde{X} , \tilde{Y} , and B are smooth complex varieties, $\dim \tilde{X} = \dim \tilde{Y} = \dim B + 1$. We suppose that

- the fibers of \tilde{q} are smooth complex curves;
- the fibers of \tilde{p} are nodal complex curves, that is, their only admissible singularities are points of transversal double selfintersection (nodes);

- a section $\gamma : B \rightarrow \tilde{Y}$ (fiberwise “infinity”) is chosen such that the “poles” of the restrictions of \tilde{f} to the fibers of \tilde{p} are smooth points of the fibers, and the orders of these poles are the same for all fibers;
- outside of the poles the family of functions is generic (in particular, if \tilde{p} has singular fibers, then their images form a subvariety of codimension 1 in B).

Denote by $Y = \tilde{Y} \setminus \gamma(B)$ and $X = \tilde{X} \setminus \tilde{f}^{-1}(Y)$ the “finite parts” of the corresponding spaces, and denote by p, q and f the restrictions of the maps \tilde{p}, \tilde{q} and \tilde{f} , respectively, to the corresponding subspaces. As a result, we obtain the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow p \quad \swarrow q & \\ & B & \end{array} \quad (5)$$

The maps in this diagram have only generic singularities. The restriction of f to a fiber of p is a holomorphic map of a nodal curve to a smooth curve, hence the space B can be treated as a family of holomorphic maps from nodal to smooth curves, and X as the universal curve over B .

The standard unfoldings of the singularity A_n given by Eq. (1) and $I_{k,\ell}$ given by Eq. (2) provide examples of such diagrams. In the source of the unfolding A_n all the fibers are smooth; each fiber is the projective line punctured at infinity. In the source of the unfolding of $I_{k,\ell}$ the fibers are smooth for $\varepsilon \neq 0$, and they are nodal for $\varepsilon = 0$; in the first case, each fiber is the projective line punctured at two points, while in the second case the fiber consists of two intersecting projective lines, each punctured at a point distinct from the point of intersection.

The requirement that the orders of all poles are the same guarantees that “the singularities of the fiberwise maps do not tend to infinity”. This means, for example, that the restriction of p to the subvariety Σ of critical points of f is proper. Therefore, for any cohomology class in X supported on Σ , its direct image in B is well defined. To be more precise, in what follows we must replace everywhere the cohomology of X by the relative cohomology $H^*(X, X \setminus \Sigma)$. But in order to keep notation simple we shall still refer to $H^*(X)$, having in mind that all the universal polynomials to be considered are supported on Σ .

Denote by $\Delta \subset X$ the subvariety of double points of p ; for a generic family B , the codimension of this subvariety is 2. Introduce the relative characteristic classes of our maps,

$$\begin{aligned} \psi &= f^*(c_1(Y)) - p^*(c_1(B)) = f^*(c_1(Y) - q^*(c_1(B))); \\ \nu &= c_1(X) - p^*(c_1(B)), \end{aligned}$$

and denote by ν_1, ν_2 the Chern roots of the normal bundle to Δ in X . The latter are well defined only on Δ ; they can be thought of as the first Chern classes of the two line bundles over Δ locally given by the tangent lines to the branches of the curve at the double point. All the formulas for the universal polynomials depend on the classes ν_1, ν_2 in a symmetric way, hence there is no need to ascribe each branch to

a specific class. More precisely, we use notation $\nu_1\nu_2 P(\nu_1 + \nu_2, \nu_1\nu_2)$ for cohomology classes representable in the form $i_*(P(c_1(\mathbf{N}_\Delta), c_2(\mathbf{N}_\Delta)))$, where $i : \Delta \rightarrow X$ is the embedding, and P is an arbitrary polynomial in the ordinary Chern classes $c_{1,2}(\mathbf{N}_\Delta)$ of the normal bundle \mathbf{N}_Δ to the embedding.

All the four cohomology classes ψ, ν, ν_1 and ν_2 can be treated both as elements of the second cohomology group $H^2(X) = H^2(X, \mathbb{Q})$, and as elements of the first Chow group of X . For the sake of definiteness, we speak below only about the cohomology ring (with rational coefficients) having in mind that all the results are valid for the Chow ring as well.

The variety X contains subvarieties $A_m(X)$ consisting of those smooth points of the fibers of p where the restriction of f to the fiber has a singularity of type A_m . Similarly, the variety Y contains subvarieties $A_{m_1, \dots, m_r}(Y)$ consisting of those points whose preimages are smooth points of the fibers of p , r of them being points where the restriction of f to the fiber has singularities of types A_{m_1}, \dots, A_{m_r} , while it is nonsingular outside these points. Denote by $[A_m(X)] \in H^*(X)$, $[A_{m_1, \dots, m_r}(Y)] \in H^*(Y)$ the cohomology classes Poincaré dual to the closures of these subvarieties. The numbers m_1, \dots, m_r form a partition, which we shall also write in the form $1^{n_1} 2^{n_2} \dots$, where n_i denotes the number of occurrences of the part i in the tuple m_1, \dots, m_r ; in particular, all but finitely many n_i are 0. By definition, $|\text{Aut}(m_1, \dots, m_r)| = n_1! n_2! \dots$.

Applying the general theory of universal polynomials for singularity classes [13] (respectively, multisingularity classes [4]) to the case under consideration [6] allows one to assert that the classes $[A_m(X)]$ (respectively, $[A_{m_1, \dots, m_r}(Y)]$) admit universal expressions in the form of homogeneous polynomials in the classes ψ, ν, ν_1, ν_2 (respectively, in the f_* -images of monomials in these classes). (The set of basic classes we have chosen here differs slightly from that in [6]. Namely, our choice there was $\Psi = \psi$, $\Sigma = \psi - \nu$, $\Delta = \nu_1\nu_2$, $N = -(\nu_1 + \nu_2)$.)

Theorem 3.1 ([6]) *There is a generating function*

$$\mathcal{R}(t_1, t_2, t_3, \dots) = \sum R_{m_1, \dots, m_r} t_{m_1} t_{m_2} \dots t_{m_r}$$

with polynomial coefficients $R_{m_1, \dots, m_r} = R_{m_1, \dots, m_r}(\psi, \nu, \nu_1, \nu_2)$ such that for any generic family of holomorphic maps of curves to curves over a base B of the form (5) having only isolated singularities the classes $[A_m(X)] \in H^*(X)$ (respectively, $[A_{m_1, \dots, m_r}(Y)] \in H^*(Y)$) coincide with the polynomials R_m (respectively, with the coefficients of $t_1 t_2 \dots t_r$ in the exponent $\exp f_*(\mathcal{R})$ of the direct image of \mathcal{R}).

Using the indeterminate coefficients method, we computed in [6] explicit expressions for universal polynomials of small codimensions. Now we are going to give formulas for all universal polynomials, of arbitrary codimension.

3.2 The formulas

The restriction of the relative cotangent bundle of p to Δ is trivial. Since all the monomials in ν_1, ν_2 arising in the universal formulas are divisible by the product $\nu_1\nu_2$

(these monomials make sense only when restricted to $\Delta = \nu_1\nu_2$), we conclude that the classes $\psi, \nu, \nu_1, \nu_2 \in H^2(X)$ introduced in the previous section are subject to the relations

$$\nu\nu_1 = \nu\nu_2 = 0.$$

Any monomial $R \in \mathbb{Q}[\psi, \nu, \nu_1, \nu_2]/\langle \nu\nu_1, \nu\nu_2 \rangle$ admits a unique representation in the form

$$R(\psi, \nu, \nu_1, \nu_2) = R_A(\psi, \nu) + R_I(\psi, \nu_1, \nu_2) - R_0(\psi),$$

where $R_A(\psi, \nu) = R(\psi, \nu, 0, 0)$, $R_0(\psi) = R_A(\psi, 0) = R_I(\psi, 0, 0)$. We call the polynomial R_0 (respectively, R_A , R_I) the 0- (respectively, A -, I -) *contribution* to R .

Being elements of the ring $\mathbb{Q}[\psi, \nu, \nu_1, \nu_2]/\langle \nu\nu_1, \nu\nu_2 \rangle$, the coefficients of the generating function \mathcal{R} whose existence is guaranteed by Theorem 3.1 split into the 0-, A -, and I -contributions, each of which, in its own turn, is a generating function:

$$\mathcal{R}(\psi, \nu, \nu_1, \nu_2; t) = \mathcal{R}_A(\psi, \nu; t) + \mathcal{R}_I(\psi, \nu_1, \nu_2; t) - \mathcal{R}_0(\psi; t).$$

The main result of the paper consists in explicit formulas for these contributions.

According to the restriction method, in order to compute the contributions explicitly, it suffices to consider “elementary blocks” in functional spaces, namely, the standard versal unfoldings of the isolated singularities (1) and (2).

Consider first the standard versal unfolding of the singularity A_n , that is, in appropriate coordinates $(x, a_2, a_3, \dots, a_{n+1})$ in X we have

$$f : (x, a_2, a_3, \dots, a_{n+1}) \mapsto (x^{n+1} + a_2x^{n-1} + \dots + a_{n+1}, a_2, a_3, \dots, a_{n+1})$$

and both maps p and q are the projections to the a -coordinates. Obviously, our four classes have the following expressions in terms of the generator τ of the ring of equivariant cohomology: $\nu = \tau$, $\psi = (n+1)\tau$, $\nu_1 = \nu_2 = 0$. Thus, each polynomial in these classes is reduced to its A -contribution.

Theorem 3.2 *For the unfolding of the singularity A_n , the generating function for the classes of multisingularities is the result of substitution $\psi = (n+1)\tau$, $\nu = \tau$ to the series $\mathcal{N}_A = \mathcal{N}_A(\psi, \nu; t)$ with rational coefficients given explicitly by the expansion*

$$\mathcal{N}_A = 1 + \psi(\psi - \nu)P_2(\nu, t) + \psi(\psi - \nu)(\psi - 2\nu)P_3(\nu, t) + \dots$$

where the coefficients P_m , $m = 2, 3, 4, \dots$, are given by

$$\begin{aligned} 1 + P_2h^2 + P_3h^3 + \dots &= \exp\left(\frac{t_1}{\nu}h^2 + \frac{t_2}{\nu}h^3 + \frac{t_3}{\nu}h^4 + \dots\right) \\ &= 1 + \frac{t_1}{\nu}h^2 + \frac{t_2}{\nu}h^3 + \left(\frac{t_1^2}{2\nu^2} + \frac{t_3}{\nu}\right)h^4 + \dots \end{aligned}$$

The terms in the expansion of \mathcal{N}_A are graded by either of the two coinciding gradings. The first grading is given by assigning degree 1 to both basic classes ψ and ν . The second one is obtained by assigning degree i to the variable t_i for $i = 1, 2, 3, \dots$.

Note that the function \mathcal{N}_A can be written conveniently as

$$\mathcal{N}_A = \exp(t_1 \nu d^2/ds^2 + t_2 \nu d^3/ds^3 + t_3 \nu d^4/ds^4 + \dots) s^{\frac{\psi}{\nu}}|_{s=1},$$

where s is an auxiliary variable (cf. [6]).

For the universal unfolding of the singularity A_n we have

$$\nu_1 = \nu_2 = 0$$

and the mapping f_* acts as follows:

$$f_* R(\psi, \nu) = (m+1)R((m+1)\tau, \tau)$$

for any polynomial R , whence

Corollary 3.3 *The A-contribution \mathcal{R}_A to the function \mathcal{R} is given by*

$$\frac{\psi}{\nu} \mathcal{R}_A(\psi, \nu; t) = \log \mathcal{N}_A(\psi, \nu; t).$$

Although the coefficients of P_n are rational rather than polynomial in ν (their denominators are powers of ν), the coefficients of $\log \mathcal{N}_A$ become polynomial when multiplied by ν/ψ , hence the coefficients of \mathcal{R}_A are polynomial:

$$\mathcal{R}_A = (\psi - \nu)t_1 - (\psi - \nu)(2\psi - \nu)t_1^2 + (\psi - \nu)(\psi - 2\nu)t_2 + \dots$$

Substituting the value $\nu = 0$ to the A-contribution, we obtain

Corollary 3.4 *The 0-contribution \mathcal{R}_0 to \mathcal{R} is given by*

$$\begin{aligned} \mathcal{R}_0(\psi; t) &= \sum (-2)^{m_1} (-3)^{m_2} \dots (2m_1 + 3m_2 + \dots - 1)_{|m|-2} \psi^{m_1+2m_2+\dots} \frac{t_1^{m_1}}{m_1!} \frac{t_2^{m_2}}{m_2!} \dots \\ &= \psi \frac{t_1}{1!} + \psi^2 \left(\frac{t_2}{1!} - 4 \frac{t_1^2}{2!} \right) + \psi^3 \left(\frac{t_3}{1!} - 6 \frac{t_1 t_2}{1! 1!} + 40 \frac{t_1^3}{3!} \right) + \dots, \end{aligned}$$

where the summation is carried over all partitions $1^{m_1} 2^{m_2} \dots$. Here $(a)_b$ denotes, for $b = -1, 0, 1, 2, \dots$, the Pochhammer symbol

$$(a)_b = a(a-1)(a-2)\dots(a-b+1) = \frac{a!}{b!}, \quad (a)_{-1} = \frac{1}{a}.$$

In order to describe the I-contribution to \mathcal{R} , let us introduce another generating function, which we denote by \mathcal{M}_A , depending on an auxiliary variable z :

$$\mathcal{M}_A(\psi, \nu; z; t) = 1 + (\psi - \nu)Q_1 + (\psi - \nu)(\psi - 2\nu)Q_2 + (\psi - \nu)(\psi - 2\nu)(\psi - 3\nu)Q_3 + \dots,$$

where the rational functions $Q_n = Q_n(\nu; z; t)$ are defined by means of the expansion

$$\begin{aligned} 1 + Q_1 h + Q_2 h^2 + \dots &= \exp \left(\frac{t_1}{\nu} h^2 + \frac{t_2}{\nu} h^3 + \frac{t_3}{\nu} h^4 + \dots \right) / (1 - zh) \\ &= 1 + zh + \left(\frac{t_1}{\nu} + z^2 \right) h^2 + \left(\frac{t_2}{\nu} + \frac{t_1}{\nu} z + z^3 \right) h^3 + \dots \end{aligned}$$

Theorem 3.5 *For any $k, \ell \geq 1$, the cohomology classes of multisingularities in the versal unfolding of the singularity $I_{k,\ell}$ are given by the generating function obtained as the result of substitution $\psi = k\ell\tau$, $\nu_1 = k\tau$, $\nu_2 = \ell\tau$ to the function $\mathcal{N}'_I + \mathcal{N}''_I$, where \mathcal{N}'_I is given by the formula*

$$\mathcal{N}'_I(\psi, \nu_1, \nu_2; t) = \mathcal{N}_A(\psi, \nu_1; t) \mathcal{N}_A(\psi, \nu_2; t)$$

and \mathcal{N}''_I is the result of replacing each monomial of the form z^n by $(n+1)t_n$ in the product

$$\psi z \mathcal{M}_A(\psi, \nu_1; z; t) \mathcal{M}_A(\psi, \nu_2; z; t).$$

Several first terms of the series \mathcal{N}'_I and \mathcal{N}''_I are

$$\begin{aligned} \mathcal{N}'_I = & 1 + \frac{\psi^2(\nu_1 + \nu_2) - 2\psi\nu_1\nu_2}{\nu_1\nu_2} t_1 \\ & + \frac{\psi^3(\nu_1 + \nu_2) - 6\psi^2\nu_1\nu_2 + 2\psi\nu_1\nu_2(\nu_1 + \nu_2)}{\nu_1\nu_2} t_2 \\ & + \frac{\psi^4(\nu_1 + \nu_2)^2 - 8\psi^3\nu_1\nu_2(\nu_1 + \nu_2) + 24\psi^2\nu_1^2\nu_2^2 - 6\psi\nu_1^2\nu_2^2(\nu_1 + \nu_2)}{2\nu_1^2\nu_2^2} t_1^2 + \dots \end{aligned}$$

and

$$\begin{aligned} \mathcal{N}''_I = & 2\psi t_1 + (6\psi^2 - 3\psi(\nu_1 + \nu_2)) t_2 \\ & + \frac{2\psi^3(\nu_1 + \nu_2) - 12\psi^2\nu_1\nu_2 + 4\psi\nu_1\nu_2(\nu_1 + \nu_2)}{\nu_1\nu_2} t_1^2 + \dots \end{aligned}$$

Similarly to the function \mathcal{N}_A , the terms of the expansions of \mathcal{N}'_I and \mathcal{N}''_I are graded by one of the two coinciding gradings. The first of them is obtained by assigning degree 1 to the classes ψ and $\nu_{1,2}$. The second one is the result of assigning degree i to the variables t_i , for $i = 1, 2, 3, \dots$

Making use of the equation $\nu = 0$ valid for the standard unfolding of $I_{k,\ell}$ and the fact that the action of f_* has the form

$$f_* R(\psi, \nu_1, \nu_2) = (k + \ell) R((k + \ell)\tau, k\tau, \ell\tau)$$

for any polynomial R , we obtain

Corollary 3.6 *The I -contribution \mathcal{R}_I to the function \mathcal{R} is given by*

$$\frac{\psi(\nu_1 + \nu_2)}{\nu_1\nu_2} \mathcal{R}_I = \log(\mathcal{N}'_I + \mathcal{N}''_I).$$

The coefficients of the function \mathcal{R}_I are polynomial rather than rational as well:

$$\begin{aligned} \mathcal{R}_I = & 1 + \psi t_1 + \psi^2(t_2 - 2t_1^2) + \dots \\ & + \nu_1\nu_2 \left(t_1^2 - t_2 + \frac{4}{3}(10(\nu_1 + \nu_2) - 3\psi)t_1^3 - 6(\nu_1 + \nu_2 - 3\psi)t_1 t_2 \right. \\ & \left. + (2(\nu_1 + \nu_2) - 5\psi)t_3 + \dots \right); \end{aligned}$$

here we separated the 0-contribution and the terms supported on Δ .

For the case of monosingularities, the general formulas look like follows.

Corollary 3.7 *The generating function*

$$\sum_{i=0}^{\infty} [A_i(X)] z^i$$

for classes of monosingularities in families with isolated singularities has the form

$$\sum_{i=0}^{\infty} [A_i(X)] z^i = L(\psi, \nu; z) + \frac{\nu_1 \nu_2}{\nu_1 + \nu_2} \left(\frac{L(\psi, \nu_1; z)}{\nu_1} + \frac{L(\psi, \nu_2; z)}{\nu_2} + (z^2 L(\psi, \nu_1; z) L(\psi, \nu_2; z))' \right) - \frac{1}{1 - \psi z},$$

where

$$L(\psi, \nu; z) = 1 + (\psi - \nu)z + (\psi - \nu)(\psi - 2\nu)z^2 + (\psi - \nu)(\psi - 2\nu)(\psi - 3\nu)z^3 + \dots$$

is the A -contribution, and $\frac{1}{1 - \psi z}$ is the 0-contribution to the corresponding classes.

4 Proofs

4.1 Proof of Theorem 3.2

The proof of Theorem 3.2 below follows, essentially, the argument in [8], see also [5]. In order to compute the cohomology class in $H^*(Y) = H_{\mathbb{C}^*}^*(\mathbb{C}^{n+1}) = \mathbb{Q}[\tau]$ dual to the stratum $A_{m_1, \dots, m_r}(Y)$, let us construct an explicit parametrization of this stratum. The points in the closure of $A_{m_1, \dots, m_r}(Y)$ admit a representation as the value and the coefficients of the polynomial

$$P(x) = \frac{1}{n+1} \int_0^x (\xi - x_1)^{m_1} \dots (\xi - x_r)^{m_r} Q(\xi) d\xi.$$

Here Q is a polynomial of an appropriate degree with leading coefficient 1, whose second coefficient is chosen so as to make the second coefficient in the expanded integrand vanish.

The explicit expression for the integral determines a \mathbb{C}^* -equivariant map $\rho : \mathbb{C}^{n+1-|m|} \rightarrow \mathbb{C}^n$, where $|m| = m_1 + \dots + m_r$ is the codimension of the stratum under consideration in Y , and the points x_1, \dots, x_r and the coefficients of Q serve as coordinates in the domain. This mapping is finite (in particular, it is proper), its image is exactly the closure of $A_{m_1, \dots, m_r}(Y)$, and the degree of the mapping is $|\text{Aut}(m_1, \dots, m_r)|$, the product of the factorials of the numbers of coinciding parts. Therefore,

$$\rho_*(1) = |\text{Aut}(m_1, \dots, m_r)| [A_{m_1, \dots, m_r}(Y)].$$

In order to compute ρ_* , note first that $\rho(0) = 0$. Considering the Poincaré dual classes, we see that $\rho(e(\mathbb{C}^{n+1-|m|})) = e(\mathbb{C}^n)$. Here $e(\cdot)$ denotes the Euler class of the normal bundle to the origin in the corresponding space. This class is equal to the product of the weights of all coordinates. Now the equations $\psi = (n+1)\tau$, $\nu = \tau$, yield

$$\begin{aligned}\rho_*(1) &= \frac{e(\mathbb{C}^n)}{e(\mathbb{C}^{n+1-|m|})} = (n+1)n(n-1)\dots(n-|m|+2)\tau^{|m|} \\ &= \frac{\psi(\psi-\nu)\dots(\psi-|m|\nu)}{\nu^{|m|}}.\end{aligned}$$

Taking these classes for the coefficients of the generating series, we obtain the function \mathcal{N}_A in Theorem 3.2. The theorem is proved.

This argument can be easily modified to construct the generating function \mathcal{M}_A for cohomology classes of multisingularities with a distinguished singular point. We shall need this generating function below in the proof of Theorem 3.5.

Consider the subvarieties $A_{i;m_1,\dots,m_r}(X)$ in the source space \mathbb{C}^{n+1} consisting of points $(x_0, a_2, \dots, a_{n+1})$ such that the polynomial

$$x^{n+1} + a_2x^{n-1} + \dots + a_{n+1}$$

has a singularity of type A_i at x_0 , (pairwise distinct) critical points of types A_{m_1}, \dots, A_{m_r} , and its value at each of these points coincide with that at x_0 . Set

$$\mathcal{M}_A(\psi, \nu; z; t_1, t_2, \dots) = \sum_{i, m_1, m_2, \dots} [A_{i;m_1,\dots,m_r}(X)] z^i t_{m_1} \dots t_{m_n}.$$

The parametrization similar to the one above proves that the function \mathcal{M}_A coincides with the function in Theorem 3.5.

4.2 Proof of Theorem 3.5

Now we want to compute the cohomology class $[A_{m_1,\dots,m_r}(Y)]$ in \mathbb{C}^* -equivariant cohomology of the versal unfolding of the singularity $I_{k,\ell}$. Denote by τ the generator of the cohomology ring $H^*(Y)$.

In contrast to the case of A_n singularity, now the subvariety $A_{m_1,\dots,m_r}(Y)$ admits no parametrization, and we must proceed differently. Instead of computing the class dual to the closure of the subvariety $A_{m_1,\dots,m_r}(Y)$, we compute, in two different ways, the intersection of this class with the hypersurface $\varepsilon = 0$ consisting of the images of the singular fibers of p . Comparing the results of the computation, we shall obtain the desired conclusion.

On one hand, we have

$$[\varepsilon = 0][A_{m_1,\dots,m_r}(Y)] = (k + \ell)\tau[A_{m_1,\dots,m_r}(Y)]$$

For the purpose of the second computation, let us split the intersection $A_{m_1,\dots,m_r}(Y) \cap \{\varepsilon = 0\}$ into irreducible components and compute their multiplicities.

As projective line degenerates acquiring a double point, the monosingularities on this line are distributed somehow between the two irreducible components of the singular curve. One of the monosingularities can find itself at the origin. The summand \mathcal{N}'_I in the I -contribution describes the situation where there is no singularity at the origin, with the distinguished critical value, while the summand \mathcal{N}''_I is in charge of such a singularity.

If the function has no singularity at the origin, then its monosingularities can be split between the two components in all possible ways, and the restriction to each component is described by the generating function \mathcal{N}_A . Each of such splittings forms an irreducible component of the intersection, and, obviously, the multiplicity of this component is 1. Replacing the class ν in two copies of the function \mathcal{N}_A by the classes ν_1 and ν_2 , respectively, and multiplying the results, we obtain the contribution of such splittings.

And if there is a monosingularity at the origin, then its restriction to each of the branches produces two local singularities. This splitting type is described by the generating function \mathcal{M}_A , whose argument z is in charge of the distinguished critical point, the one at the origin. Two copies of this function give the product

$$\mathcal{M}_A(\psi, \nu_1; z; t) \mathcal{M}_A(\psi, \nu_2; z; t).$$

A function having local singularity x^a on one branch of the curve at the origin and y^b on the other branch, can result from a degeneration of an A_m singularity on the smoothened curve iff $a + b = m + 1$. Any such partition (a, b) of $m + 1$ determines an irreducible component of the local singularity. The multiplicity of intersection of such a component with the divisor $\{\varepsilon = 0\}$ is independent of a and b and equals $m + 1$. Indeed, this multiplicity coincides with the dimension of the local algebra $\mathbb{C}[x, y]/\langle x^a + y^b, xy \rangle$, which is $a + b = m + 1$. Therefore, the multiplicities of all the components are the same, and in order to compute their contribution one must multiply the product of the generating functions \mathcal{M}_A by ψz and replace in the result each monomial z^{m+1} by $(m + 1)t_{m+1}$. This completes the proof of the theorem.

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